An improved construction algorithm of polar codes based on the frozen bits*

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In order to improve the problems that the minimum hamming weight (MHW) of the polar codes of the traditional Gaussian approximation (GA) construction is small and its performance is not good enough, an improved channel construction algorithm of polar codes based on frozen bits is proposed by combining the construction of the Reed-Muller (RM) code to effectively increase the MHW and analyzing the correcting and checking functions of the frozen bits in the successive cancellation list (SCL) decoding. The construction algorithm selects the channel with the smaller row weight corresponding to the information channel in the channel construction stage, and some channels are set as the frozen channels under the proposed frozen channel setting principle. So the proposed construction algorithm not only eliminates the channels with the smaller row weight and optimizes the distance spectrum of polar codes, but also makes full use of the checking ability of the frozen bit in SCL decoding to improve the error correction performance of polar codes. The polar codes constructed by this algorithm are named as FRM-polar codes. The simulation results show that the proposed FRM-polar codes have a larger performance gain than the RM-polar codes and the polar codes constructed by GA under different code-lengths. In addition, the proposed construction algorithm has the same complexity as the construction algorithm of the RM-polar codes.

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Polar codes have been proven to be the first error-correcting codes that can achieve the channel capacity under the binary memoryless symmetric channels[1], and have been selected as the coding scheme for the control channels in the 5G standard, which is a current research hotspot in the field of channel coding[2,3].

Polar codes share some traits with both algebraic codes and probabilistic codes because they were developed at the same time. The channel with the maximum conditional mutual information is chosen as information channels in the construction stage, resulting in a smaller minimum hamming weight (MHW) for polar codes and still worse decoding performance than turbo or low density parity check (LDPC) codes[4]. In this regard, researchers have worked on the improvement of the distance spectrum. Cascading additional codes and modifying the choice of the information channels and the frozen channels are the two basic ways to improve the distance spectrum. In the first case, one approach is to place the parity check (PC) codes in the channels corresponding to the rows of the generation matrix with smaller weights[5] and a new code design algorithm aiming at minimizing the decoding error probabilities of successive cancellation (SC) decoding and maximum likelihood (ML) is proposed in Ref.[6]. Besides, Ref.[7] and Ref.[8] further increase the minimum hamming distance of the cyclic redundancy check aided polar (CRC-polar) codes. In the second case, polar codes are designed by the construction of polarized weight (PW) in fading channels[9,10]. In addition, a hybrid code combining Reed-Muller (RM) codes and polar codes construction is introduced in Ref.[11] and Ref.[12], called RM-polar codes. RM codes and polar codes are generated from the same matrix but different subsets of rows. RM codes select only the rows with maximum weight, so RM-polar codes have much larger MHW and better error correction performance than polar codes. In Ref.[13], a new improved RM-polar code and permutation selection scheme for belief propagation list (BPL) decoding are proposed by a special permutation of factor graph. This construction algorithm is optimal for BPL decoding, but it still has to be polished for the successive cancellation list (SCL) decoding.

Through analyzing the encoding and decoding process of polar codes, it can be concluded that the error-correction ability of polar codes mainly comes from the frozen bits. The placements of the frozen bits

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will have a considerable impact, especially for the SCL decoding which must employ the output of the prior decoding, and they should be carefully studied.

For improving the performance of the SCL decoding, there are several coding and decoding schemes relevant to frozen bits proposed in Refs.[14—16]. On the role and positioning of frozen bits in the SCL decoding, there is, however, a dearth of studies. In order to solve this issue, a channel construction algorithm that is better suited for the SCL decoding algorithms is provided by integrating the creation of RM codes with analysis of the error correction and check functions of frozen bits in the SCL decoding algorithms.

For a polar code of code length $N=2^n$, its encoding can be achieved by generating matrix $\boldsymbol{G}_N$, where $\boldsymbol{G}_N=\boldsymbol{B}_N\boldsymbol{F}^{\otimes n}$, $\boldsymbol{B}_N$ is the bit-inverse permutation matrix and $\boldsymbol{F}^{\otimes n}$ is the nth Kronecker product of $\boldsymbol{F}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$. An RM code and an ($N$, K) polar code both can be generated as

$$\boldsymbol{x}_1^N=\boldsymbol{u}_1^N\boldsymbol{G}_N, \tag{1}$$

where K is the number of information bits, the input sequence is $\boldsymbol{u}_1^N=\{u_1,u_2,...u_N\}$ and the output sequence is $\boldsymbol{x}_1^N=\{x_1,x_2,...x_N\}$. The input sequence $\boldsymbol{u}_1^N$ consists of two parts: information sequence $\boldsymbol{u}_A$ used to transmit the information bits, and frozen sequence $\boldsymbol{u}_{A^c}$ agreed to be an all-0 sequence for convenience.

Although both RM codes and the conventional polar codes are generated by selecting a subset of rows from the same matrix $\boldsymbol{G}_N$, the channel selection way of the polar codes is different and is based on the conditional mutual information, accompanied by smaller row weights. The MHW of the polar codes is equal to the minimum weight of rows corresponding to the information channels[8], and smaller MHW leads to performance degradation, while RM codes select information channels with larger weights but lower reliability, which also leads to poor performance.

In Ref.[11], a new RM-polar code is constructed by exchanging the information channel index of the polar codes and the information channel index of the RM codes selectively, which can remove the channels with low channel reliability and low row weight. A (N, K, f) RM-polar code is constructed as follows, where N is the code length, K is the number of information bits, and f is the number of swapped bit indexes.

First, select the first (K+f) channels in descending order of reliability obtained by the estimation method of Gaussian approximation (GA). Among the selected channels, find the channels with lowest row weight and the number is A. If $A\geq f$, the channels with the lowest reliability of number f are set as frozen channels. If $A<f$, find the channels with the second lowest row weight, and then number A channels with the lowest row weight and number (f−A) channels with the second lowest row weight and less reliability are set as the frozen channels.

This section will analyze the function of frozen bits from two perspectives of polar codes: the SC decoding and the SCL decoding.

The SC decoding algorithm is the most basic decoding algorithm for polar codes with the feature that decoding the current bit uses all the previous decoded bits. Since all frozen bits are set to a fixed value, even when the regular log likelihood ratio (LLR) decoding result is not match with it, the current frozen bit still can be directly decoded as the preset value and thus the mistake is corrected. Other than that, the error-correction function of the frozen bits is also reflected in the later decoding bits because they will use the decoding result of the current bit. It infers that the placement of the frozen channel has a partial influence on the decoding performance.

The SCL decoding algorithm performs path expansion at each information bit decoding and keeps L possible candidate paths at each stage of path selection to increase the probability that the correct path can be preserved to the end. Then the path with the smallest path metrics (PM) value is selected from the L candidate paths as the decoding result finally, which effectively avoids the error propagation phenomenon.

In the SCL decoding, when the frozen bit verdict does not match the LLR verdict, a PM value will be added to the corresponding path causing the path may be deleted. There are two cases that lead to incorrect judgments of frozen bits: the first one is the effect of noise and the second one is the error of previous decoded bits. The noise is random and uncontrollable, so the effect of it is not considered. For the second case, the ideal state of the SCL decoding process contains 1 correct path and (L−1) error paths. The error probability of LLR verdict is higher when the error path is decoded to the frozen bit, and the probability that the correct path is kept until to the end is increased by clipping the paths with larger PM value. Therefore, the frozen bits also have a portion of the check capability.

The above analysis shows that the frozen bits have error correction and check function in the SCL decoding. Considering the frozen bit as an error correction and check code, the placement and number of the frozen bits should be further studied.

Only the position of frozen bits is investigated in this work because the code length and code rate dictate the number of frozen bits. The reliability estimation method used in this study is the GA, and the signal noise ratio (SNR) is fixed at 2.5 dB since SNR has an impact on the GA-built channel. The reliability metric is calculated as

$$m_{2N}^{(2i-1)}=\varphi^{-1}(1-[1-\varphi(m_N^{(i)})]^2), \tag{2}$$

$$m_{2N}^{(2i)}=2m_N^{(i)}, \tag{3}$$

$$m_1^{(1)}=2/\sigma^2, \tag{4}$$

$$p(A_i)=\int_{-\infty}^{0}\frac{1}{2\sqrt{\pi m_N^{(i)}}}\cdot\exp\left(\frac{-(x-m_N^{(i)})^2}{4m_N^{(i)}}\right)\mathrm{d}x, \tag{5}$$

where $m_N^{(i)}$ denotes the average value of LLR, and $p(A_i)$

is the channel reliability metric. In order to study the error correction function of frozen bits separately, SC decoding is used for decoding. Taking the (1 024, 512) polar code as an example to study the effect of frozen bit positions on the error-correction performance. Four channels with similar reliability are selected from the 19 channels with the same row weight of 16 among all information channels: the 737th, 837th, 809th, and 835th channels with reliability measures of 26.542 5, 26.541 4, 24.193 1, and 24.190 8, respectively. Each of the four channels is set up as a frozen channel. The error-correction performance comparison is illustrated in Fig.1.

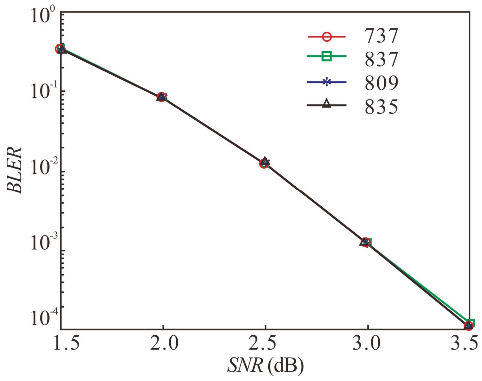

Fig.1 Block error rate (*BLER*) performance of SC decoding when each of the four channels is set as a frozen channel

Fig.1 shows that the performances of the four frozen channels are similar, which is consistent with the conventional construction of polar codes. However, the error-correction performance of the frozen bits is superimposable, and the total error-correction performance for the entire polar code is determined by the number of frozen bits, even though the error-correction performance may differ for one specific channel due to the different positions of the frozen bits. Position has very little impact on performance.

The distribution of (1 024,512) polar code channels is listed as shown in Tab.1, which illustrates the distribution of frozen channels in various channel segments, in order to analyze the check function of frozen bits in the SCL decoding. It is evident that there are less frozen channels in the later channel segments. The channel segment between channels 897th and 1 024th, for instance, has just one frozen channel out of a total of 128 channels, which has a very poor check capacity and a lower error-correction performance.

Next, the 737th, 837th, 809th, and 835th channels are each set as a frozen channel in turn. The error-correction performance is shown in Fig.2. When the two channels have the same row weight and similar dependability, there is a more pronounced performance discrepancy. Performance is improved when the frozen channel is set back, and the performance gap widens as the channel distance increases. As a result of the distribution of channel segment study, the 737th channel is a part of the sixth channel segment which has the most frozen channels, and its error-correction performance gain after adding a frozen channel is minimal due to the check capacity being nearly saturated. The 837th channel belongs to the seventh channel segment, which has less frozen channels, hence the performance improvement is greater with its addition. However, the 809th and 835th channels both belong to the 7th segment and have lesser performances boost, indicating that the precise location of frozen bits needs to be analyzed in conjunction with the original frozen channels more thoroughly. It is important to note that the segmentation in the form of eight evenly spaced segments is inaccurate.

Tab.1 Number of channels for the (1 024, 512) polar code segmentation

Channel segment	Number of information channels	Number of frozen channels
1—128	1	127
129—256	18	110
257—384	28	100
385—512	88	40
513—640	42	86
641—768	100	28
768—896	108	20
897—1 024	127	1

The first frozen channel setting principle, which states that various channel segments should have the appropriate number of frozen channels established, may be inferred from the analysis above.

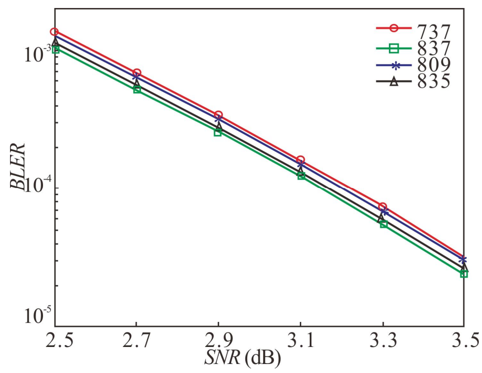

Fig.2 *BLER* performance of SCL decoding when each of the four channels is set as a frozen channel

Additionally, Fig.2 shows that although the distance between the 835th and 837th channels is lower when they are set up as frozen channels, respectively, the performance gap is significant. The distribution of the original frozen channels reveals that the 833rd channel is an original frozen channel. Due to the close proximity to 834th channel, while setting the 835th channel as a frozen channel, the check capability of the frozen bit cannot be completely utilized, makingthe performance benefit less noticeable. Surprisingly, putting the 837th channel as a frozen channel improves performance even though it has a larger reliability measure. This shows that when creating a frozen channel, it is important to take location and the distance to the original frozen channels into account as well.

The conventional configuration places frozen channels at the front position primarily because it is the channel with the lowest reliability, which will result in few protected bits and a limited ability to correct errors. The investigation that follows is carried out in this paper to examine the effects of distance and reliability. The 833rd and 897th channels, for example, are two original frozen channels. An information channel segment from the 834th channel to the 896th channel has a length of 63 and consists of six channels with row weights of 16. These channels are the 834th, 835th, 837th, 841st, 849th, and 865th channels. The distances between each of them and the 833rd channel are 1, 2, 4, 8, 16, and 32, respectively, and their related reliability measures are 21.889 7, 24.190 8, 26.541 4, 28.931 7, 31.354 1, and 33.803 0. The performance is displayed in Fig.3 with every one of them set as a frozen channel.

Even if the 865th channel has the highest reliability, Fig.3 demonstrates that it performs best when set to the frozen channel, which goes against the conventional wisdom that the frozen channels are the least reliable channels when SCL decoding with polar codes is used. Due to the fact that the first three bits of the R1 node-namely, the 834th, 835th, and 836th channels have the highest error probability, the performance gap is also most obvious for the shorter lengths of the 835th channel and 837th channel. With the highest mistake probability, these three places can be verified by freezing the 837th channel. The performance increase is not significant since the reduced performance gap that results from setting each of the following three channels to frozen is caused by the lower mistake probability at the later places. However, at this point, performance improvement still outweighs performance decrease brought on by reliability degradation.

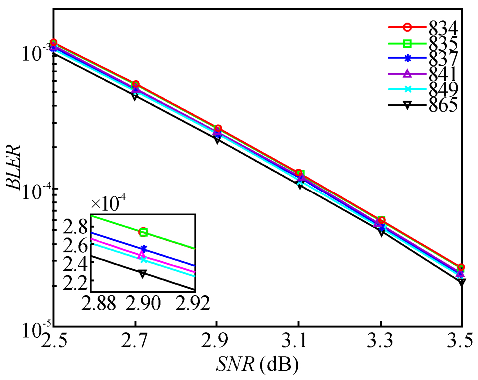

Fig.3 *BLER* performance of SCL decoding when each of the six channels is set as a frozen channel

From the above analysis, the second frozen channel setting principle can be gotten, that is, it is preferable to set the frozen channel in the information channel segment at the backward location with the smallest row weight.

According to the research above, while choosing frozen channels, it is important to consider their connections with other frozen channels in order to fully utilize the frozen bits and enhance the polar codes' error-correction and check capabilities. By using the frozen bits as a check code and integrating the row weight selection criteria of RM codes with the reliability selection rules of polar codes, an improved channel construction algorithm for polar codes is proposed. The following is the approach for setting frozen channels that is suggested in this paper.

The proper number of bits of frozen channels should be configured for channel segments with varying reliability. Long information channel segments should be set up with frozen channels at intervals, and the frozen channels should be set up in the backward position which has the lowest row weight.

Tab.2 shows the improved construction algorithm of polar codes based on the frozen bits in this paper.

Tab.2 An improved construction algorithm of polar codes based on the frozen bits

Algorithm 1 Improved construction algorithm
Input: N, K, f
output: A^c
1: The reliability is obtained by the GA construction, and the (K+f) channels with the highest reliability are selected;
2: Confirm $\lvert A_m \rvert$ and $\lvert A_s \rvert$ through Eq.(7) and Eq.(8);
3: if $f<\lvert A_m \rvert$
4: The f channels in A_m are selected as frozen channels according to the proposed setting algorithm, and the rest of the channels are information channels;
5: else
6: Selecting all channels in A_m and ($f-\lvert A_m \rvert$) channels in A_s selected according to the proposed algorithm as frozen channels, and the rest of the channels are information channels.
7: end

Before proceeding with the selection of the number and position of frozen bits, the definition of Eqs.(6), (7) and (8) is required.

$$f=\log_2 N\times(\alpha-\lvert K/N-1/2\rvert^2), \qquad (6)$$

where f is the number of exchanged frozen channels and if f is not an integer, it is rounded off. α is a factor to flexibly adjust the number of frozen bits, and $0<\alpha<1$ usually.

$$A_m=\{i\notin A^c \mid w(g_n^{(i)})=d_m\}, \qquad (7)$$

$$A_s=\{i\notin A^c \mid w(g_n^{(i)})=d_s\}, \qquad (8)$$

where d_m and d_s in Eq.(7) and Eq.(8) are the MHW and sub-MHW of the set of non-frozen bits, and $w(g_n^{(i)})$ is the hamming weight value of the ith row in $\boldsymbol{G}_N$. The frozen bits are selected from the sets A_{m} and A_{s}. The algorithm of frozen bits position selection in this paper is described as Algorithm 1 in Tab.2.

Taking N=1 024, K=512, f=8 as an example, the 520 channels with the largest reliability measure are selected firstly, and then 19 channels with the smallest row weight are selected among them: 713th, 721st, 737th, 805th, 809th, 817th, 834th, 835th, 837th, 841st, 849th, 865th, 898th, 899th, 901st, 905th, 913th, 929th, 961st channels. The distribution of the initial frozen channels is shown in Tab.3.

Tab.3 Channel distribution of (1 024, 512) polar codes

Channel segment	Frozen channel distribution
709—799	709, 769—775, 777—779, 781, 785—789, 793
800—895	801—803, 833
896—1 024	896

The three lengthy information channel segments (710th—768th; 834th—895th; and 897th—1 024th) are displayed in Tab.3 and should set frozen channels at regular intervals. There is no need to add more frozen channels in the first segment due to already 18 frozen channels. The 865th, 849th, and 841st channels are chosen in the second segment which has 4 frozen channels. There should be more frozen channels set since the third segment only has one frozen channel, so the 961st, 929th, 913th, 905th, and 901st channel are all chosen. The three channel segments now each have 18 frozen channels, 7 frozen channels, and 6 frozen channels, respectively. This distribution offers the best performance when compared to other distributions. This coincidence that these 8 channels happen to be the 8 most reliable among the 19 channels can be attributed to the polar codes' unique structure, i.e., the reliability and row weights are incremented in segments. The suggested frozen channels setting approach is therefore roughly similar to choosing the channels with the best reliability from all of the channels with the lowest row weight.

In order to verify the superiority of the performance of the proposed improved polar codes channel construction algorithm based on frozen bits, the simulation comparison analysis among the proposed construction algorithm, the GA construction algorithm and the construction algorithm of the RM-polar code is performed in this paper, and the newly constructed polar codes are named as the FRM-polar codes in this paper. All the simulations are performed over additive white Gaussion noise (AWGN) channels with binary phase shift keying (BPSK) modulations in this paper. For the selection of f, when f is larger, the constructed distance spectrum can be optimized better, but the channel's dependability suffers more. As a result, the A_{m} is larger and the f selection is larger, when the code length is longer. Fig.4 and Fig.5 depict the SCL decoding with lists size of L=8 and a comparison of the error-correction performance attained in four different code length scenarios.

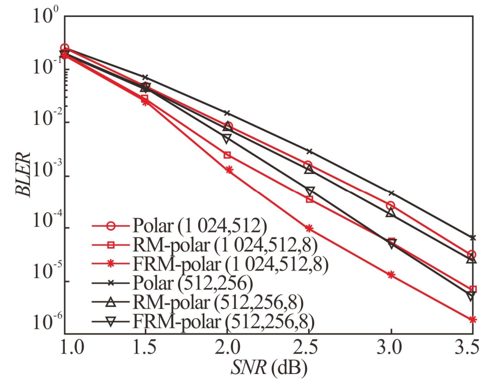

Fig.4 *BLER* performance of SCL decoding of three polar codes at long code

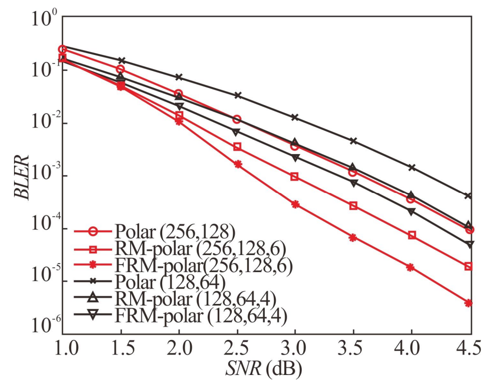

Fig.5 *BLER* performance of SCL decoding of three polar codes at short code

As observed in Fig.4, at $BLER$=10^{-4}, the (1 024, 512, 8) FRM-polar codes proposed in this paper have gains of approximately 0.35 dB and 0.75 dB compared to the RM-polar codes in Ref.[13] and conventional GA constructed polar codes, and (512, 256, 8) FRM-polar codes have gains about 0.3 dB and 0.55 dB over RM-polar codes and GA constructed polar codes, respectively. As can be seen from Fig.5, at $BLER$=10^{-3}, the proposed (256, 128, 6) FRM-polar codes have gains about 0.35 dB and 0.9 dB than the RM-polar codes and GA constructed polar codes, and the (128, 64, 4) FRM-polar codes have gains about 0.3 dB and 0.8 dB than RM-polar code and GA constructed polar codes, respectively. The FRM-polar codes reduce some of the channel reliability, but improve the error correct and check function of the SCL decoding and its decoding performance. In conclude, at various code lengths, the proposed FRM-polar codes outperform the RM-polar codes and the GA constructed polar codes.

The complexity of the proposed FRM-polar codes and RM-polar codes construction algorithm is compared and analyzed.

The proposed FRM-polar code construction algorithm can be simplified as follows. Based on the GA construction, select the f channels with the highest reliability among those with the lowest row weight as the frozen channels. The frozen channels for the RM-polar codes are the f channels with the smaller reliability out of all the channels with the lowest row weight. The additional operations of both algorithms are the ranking of row weights and the ranking of channel reliability. The proposed FRM-polar construction algorithm and the RM-polar code construction algorithm are both channel reliability ranking problems with fixed row weights, so the complexity of both algorithms is the same.

In this paper, the function of frozen bits in the SC decoding and the SCL decoding and the disadvantages of the GA construction algorithm are analyzed. An improved polar codes construction algorithm based on frozen bits is proposed in conjunction with the construction method of RM codes. The simulation demonstrates that this construction algorithm performs significantly better than the GA construction algorithm and the RM-polar code construction algorithm while no increase in complexity. The proposed construction algorithm breaks the mindset of the conventional construction algorithm according to the reliability and offers a fresh idea for the investigation direction of polar codes channel construction by considering the distance and interconnection between channels, weighing the benefits and drawbacks of channel reliability and channel distance selection.

Ethics declarations

Conflicts of interest

The authors declare no conflict of interest.

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