Simplified SCL decoding algorithm of polar codes based on critical sets*

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In order to reduce the high complexity of the successive cancellation list (SCL) algorithm for polar codes, a simplified SCL decoding algorithm based on critical sets (CS-SCL decoding algorithm) is proposed. The algorithm firstly constructs the critical sets according to the channel characteristics of the polar codes as well as comprehensively considering both the minimum Hamming weight (MHW) of the information bits and the channel reliability. The information bits within the critical sets and the path splitting are still performed by the SCL decoding algorithm while the information bits outside the critical sets are directly performed by the hard decision. Thus, the number of path ordering, copying, and deleting can be reduced during decoding. Furthermore, the computational complexity of the SCL decoding can also be reduced. Simulation results demonstrate that the decoding complexity of the proposed CS-SCL decoding algorithm, compared with the conventional SCL decoding algorithm, is reduced by at least 70%, while compared with the simplified SCL (PS-SS-SCL) algorithm which constructs the critical set with the first and second information bits of the Rate-1 nodes, its decoding complexity can also be reduced. Moreover, the loss of the error correction performance for the proposed CS-SCL decoding algorithm is minor. Therefore, the proposed CS-SCL algorithm is effective and can provide a reasonable tradeoff between the decoding performance and complexity for the decoding algorithm of polar codes.

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Polar codes have been proven to be the first codes that can achieve the channel capacity under the symmetric binary input memoryless channels[1]. Furthermore, polar codes have successfully been selected as the coding scheme of the control channel for the enhanced Mobile BroadBand (eMBB) in the 5G standard and play a significantly important role in next-generation mobile communication systems.

Unfortunately, when the code length is finite, the decoding performance of the successive cancellation (SC) algorithm is not satisfactory due to incomplete channel polarization and its computational complexity is $O(N\log N)$[2,3], where N is the code length. Hence, the successive cancellation list (SCL) algorithm introduced in Ref.[4] has improved the decoding performance close to that of the maximum likelihood (ML) decoder at high signal-to-noise ratio (*SNR*) by preserving multiple splitting paths. But the computational complexity of the SCL decoder rises to $O(LN\log N)$ because of operations such as copying, ordering, and deleting (where L is the list size parameter), which makes it difficult to be applied in practice when L is very large. To further improve the decoding performance, the cyclic redundancy check aided SCL (CA-SCL) decoding algorithm was proposed in Ref.[5] and Ref.[6] to save more memory space and decoding delay. The parity check (PC) aided CA-SCL coding and decoding algorithm can reduce the computational complexity and improve the decoding performance[7]. In a word, it is still meaningful to reduce the computational complexity of the SCL algorithm. Actually, the SCL decoder preserves two decoding results of "0" and "1" for each information bit and each decoding path will be split into two paths until the end of the decoding procedure, but the optimal decoding path remains L, increasing the computational complexity of $O(LN\log N)$. Recently, numerous more efforts have been made to reduce the computational complexity from the perspective of reducing the number of splitting times. Ref.[8] reported the segmented list-pruning (SLP) algorithm and the modified distributed sorting (MDS) algorithm to reduce the list size and complexity of sorting operations. In Ref.[9], a hypothesis-testing-based strategy was designed to select reliable unstructured nodes for hard decision. Ref.[10] used loglikelihood ratio (*LLR*)

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values to decide whether the corresponding bits were reliable enough not to split paths. Ref.[11] constructed a critical set consisting of the first information bit of Rate-1 nodes (or the first and second information bits when L is large). Path splitting performs only for the information bits within the critical set, but there is a significant loss in error correction performance under the channel reliability evaluation method of polarization weight (PW) in the 5G standard. In Ref.[12], the critical set was utilized in the node-wise SCL decoding progress to lower the decoding latency. In comparison to the above-mentioned techniques, the critical set is more amenable to hardware implementation because it may be acquired offline and does not need to be determined through the decoding process. Nevertheless, the techniques in Ref.[11] and Ref.[12] considered only the reliability of the polarized channels but not the error probability of decoding the information bits with the minimum Humming weight (MHW).

For the above problems, a simplified SCL decoding algorithm based on critical sets (CS-SCL) with low complexity and small loss is proposed to improve the decoding performance of polar codes in this paper. In the proposed algorithm, both the MHW of the information bits and the channel reliability are considered and the channel reliability is evaluated by the PW.

Consider a polar code of code length $N=2^n$, where the input source sequence is denoted by u_1^N. The codeword sequence is obtained by $x_1^N=u_1^N\boldsymbol{G}_N$, where $\boldsymbol{G}_N=\boldsymbol{F}^{\otimes n}$ is the generation matrix, as the n-th Kronecker power of $\boldsymbol{F}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$. Then the codeword x_1^N is modulated and transmitted in the channel, and the received sequence is denoted by y_1^N.

Let W denote a binary input memoryless channel, with input $\boldsymbol{X}\in\{0,1\}$, output $\boldsymbol{Y}$, and the channel transition probability $W(y|x)$, where $x\in\boldsymbol{X}=\{0,1\}$ and $y\in\boldsymbol{Y}$. After the effect of channel polarization, the LLR of the i-th sub-channel $W_N^{(i)}(y_1^N,u_1^{i-1}|u_i)$ is defined by

$$L_N^{(i)}(y_1^N,u_1^{i-1}|u_i)=\ln\frac{w_N^i(y_1^N,u_1^{i-1}|u_i=0)}{w_N^i(y_1^N,u_1^{i-1}|u_i=1)}. \quad (1)$$

The set of unfrozen bits of polar codes is denoted by $\boldsymbol{A}$. The set of frozen bits is denoted by $\boldsymbol{A}^{\mathrm{c}}$ and their values are typically zero. The SC decoder performs hard decision by regarding the LLR of every information bit $u_i\in\boldsymbol{A}$ as the decoding path metric (PM) and executes bit-by-bit decoding according to

$$u_i=\begin{cases}0, LLR>0\\ 1, LLR\le 0\end{cases}. \quad (2)$$

For short and moderate code lengths, since the channel polarization is incomplete, the errors in decoding information bits occur and the error propagation arises, and the block error rate ($BLER$) of the SC decoding algorithm rises sharply.

The SCL decoder performs path splitting when decoding each unfrozen bit and preserves both "0" and "1" decoding results. The PM[13] of each path is calculated as

$$PM_i^{(\ell)}=\sum_{j=0}^{i}\ln\left(1+\mathrm{e}^{-(1-2\hat{u}_j[\ell])\cdot L_N^{(j)}[\ell]}\right). \quad (3)$$

The larger the PM value, the smaller the probability of the correction for the corresponding decoded path.

To facilitate the hardware implementation in practical applications, the PM can be simplified by

$$PM_i^{(\ell)}=\begin{cases}PM_{i-1}^{(\ell)}, & \hat{u}_i[\ell]=\frac{1}{2}\left(1-\mathrm{sgn}\left(L_N^{(i)}[\ell]\right)\right)\\ PM_{i-1}^{(\ell)}+\left|L_N^{(i)}[\ell]\right|, & \text{else}\end{cases}, \quad (4)$$

where $PM_l^{(i)}$ denotes the PM after decoding u_i on the l-th path. $\hat{u}_i[\ell]$ and $L_N^{(i)}[\ell]$ represent the estimated value of u_i and the LLR for the l-th path, respectively.

When the number of splitting paths exceeds the maximum number of decoding list size L, only L paths with the smaller PM is survived. After the last bit is decoded, the path with the smallest PM is selected from the paths that pass the CRC check and outputs as the codeword of the CA-SCL decoder.

If the channel corresponding to the unfrozen bit is reliable enough, although the SCL decoder retains only one of the decoding results from "0" or "1" by Eq.(2) and the original L paths don't split into $2L$ paths, the number of candidate paths can be reduced and there is still no significant loss in the error performance of polar codes. As such, the SCL decoder (with the list size of L) only needs to update its decoding result without picking the L paths with smaller PM values from the $2L$ candidate paths, and hence the copying, sorting, and deleting of paths can be avoided during the picking process in decoding, resulting in further complexity reduction. For example, when decoding information bits u_1 to u_4, if the channels that transmit u_1 and u_2 are unreliable and yet the channels that transmit u_3 and u_4 are reliable, the decoding tree of the SCL algorithm (with list size L=4) for path reduction is shown in Fig.1.

In Fig.1, the SCL decoder preserves two decoding results of u_1 and u_2, and one decoding result of u_3 and u_4. Since the decoding results of u_1 and u_2 preserved by each SC decoder are different, the estimated values of u_3 and u_4 in decoding are also different. When u_3 or u_4 is decoded, L candidate paths and one process of path copying, sorting, and deleting are concurrently eliminated, reducing the computational complexity of the SCL decoding algorithm.

The communication quality of the channel can be considered by the factors of the sub-channel reliability, Hamming weight, and special nodes. Among the channels corresponding to unfrozen bits, the channels with comprehensively lower reliability lead to degradation of the error performance for the polar codes, so their index need to be selected into the critical set, and both the decoding results of "0" and "1" are retained. By contrast, the channels with high quality can still keep the original high error correction performance although one result of "0" or "1" is retained, so their index doesn't need to be selected into the critical set. The channels corresponding to frozen bits remain unchanged and the bits are set to "0" by default.

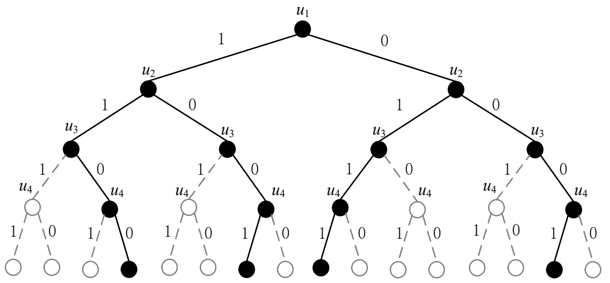

Fig.1 Decoding tree of SCL decoding for paths reduction

The explicit expression of the union bound on the error probability of subchannels based on the polar spectrum is given in Ref.[14]. Given the code length N, the i-th subcode $\mathbb{C}_N^{(i)}$ is defined as a set of codewords

$$\mathbb{C}_N^{(i)} \triangleq \left\{c : c = (0_1^{(i-1)}, u_i^N)\boldsymbol{G}_N, \forall u_i^N \in \{0,1\}^i\right\}. \tag{5}$$

One subset of the subcode $\mathbb{C}_N^{(i)}$, namely the polar subcode $\mathbb{D}_N^{(i)}$, can be defined as

$$\mathbb{D}_N^{(i)} \triangleq \left\{c : c = (0_1^{(i-1)}, 1, u_{i+1}^N)\boldsymbol{G}_N, \forall u_{i+1}^N \in \{0,1\}^i\right\}. \tag{6}$$

The polar spectrum of the polar subcode $\mathbb{D}_N^{(i)}$ also named as polar weight distribution, is defined as the weight distribution set $\left\{A_N^{(i)}(d)\right\}, d \in [\![1, N]\!]$, where d is the Hamming weight of non-zero codeword and the polar weight enumerator $A_N^{(i)}(d)$ enumerates the codewords of weight d for codebook $\mathbb{D}_N^{(i)}$.

The error probability of the i-th subchannel $W_N^{(i)}$ is further upper bounded by

$$P\left(W_N^{(i)}\right) \le \sum_{d=1}^{N} A_N^{(i)}(d) P_N^{(i)}(d), \tag{7}$$

where the pairwise error probability $P_N^{(i)}(d)$ is determined by the Hamming weight d of the codeword c for $\mathbb{D}_N^{(i)}$. And MHW for $\mathbb{D}_N^{(i)}$ equals to the Hamming weight of the i-th row in the generation matrix $\boldsymbol{G}_N$.

The union bound on the error probability of subchannels intuitively indicates that the Hamming weight affects the pairwise error probability while the weight enumerator determines the number of the corresponding error events.

The error numbers of bits, as shown in Fig.2, is obtained after 10 million simulations by transmitting all "0" input sequences. The simulated polar codes with length N=1 024, code rate R=1/2, and list size L=8, take $g(x)=x^{16}+x^{12}+x^6+x^5+x+1$ as the CRC generator polynomial.

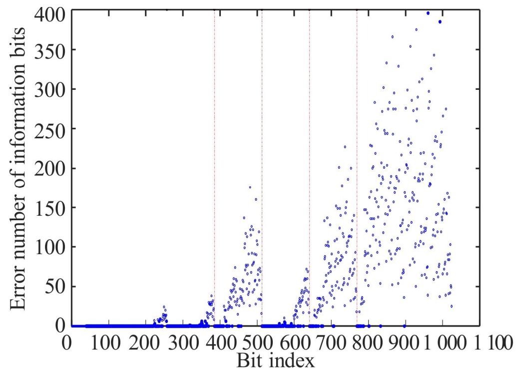

Fig.2 Error number of information bits

Fig.2 demonstrates that the error number of information bits can be divided into six parts with a stepped distribution, and the most error-counted bits of each sub-polar code always correspond to the bits with MHW. Therefore, the polar code can be decomposed into six sub-polar codes, as shown in Fig.3. The code length of the first sub-polar code and the last sub-polar code is $N = 2^{n-2}$, which corresponds to the smallest and largest part of the bit index respectively, and the remaining 4 sub-polar codes with code length $N = 2^{n-3}$ are between the two longer sub-polar codes.

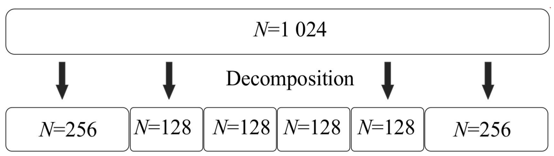

Fig.3 Decomposition of polar codes into sub-polar codes

The set of the MHW and the set of the subminimum HMW can be defined as

$$\boldsymbol{A}_m = \{i \notin \boldsymbol{A}^c \mid w(g_n^{(i)}) = d_m\}, \tag{8}$$

$$\boldsymbol{A}_s = \{i \notin \boldsymbol{A}^c \mid w(g_n^{(i)}) = d_s\}, \tag{9}$$

where the Hamming weight of the i-th row in the generation matrix $\boldsymbol{G}_N$ is the number of 1. The d_m and d_s in Eq.(8) and Eq.(9) represent the MHW and sub-MHW of the unfrozen bits set, respectively.

Thus, select the index of MHW bits in each sub-polar code as shown in Eq.(8) to construct the preliminary critical set. If the number of bits with MHW in the sub-polar code is less than 1% of the number of bits in the unfrozen bit set (rounded off), the index of sub-MHW bits for the sub-polar code as shown in Eq.(9) is added to the critical set.

The PW method proposed in Ref.[15] is taken in this paper for measuring the reliability of the polarized channels. For a given polarized channel with index i, where the binary expansion of i is $B=(b_{n-1},\dots,b_1,b_0)$, the PW value corresponding to channel i is defined as

$$W_i \triangleq \sum_{j=0}^{n-1} b_j \beta^j, \tag{10}$$

where the value of β is $2^{1/4}$.

The channel reliability can be ranked through the PW of polarized channels calculated by Eq.(10). Fig.4 shows the reliability of polarized channels for code length N=1 024, where the bit index with the value of PW equaling 0 indicates that the channel transmits frozen bits, while bit index with positive PW value indicates that the channel transmits unfrozen bits, and a larger PW value indicates that the channel is more reliable.

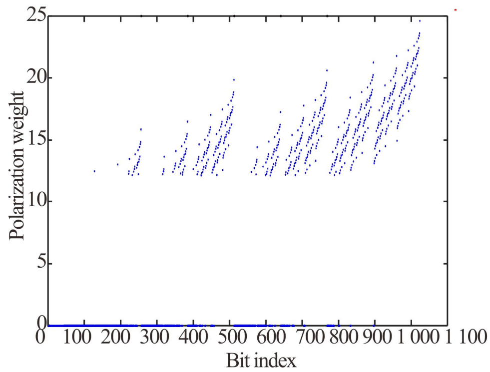

Fig.4 Reliability of polarized channels evaluated by polarization weights

By analyzing Fig.2 and Fig.4, it can be seen that the channel with a low PW value has a lower reliability ranking in the set of unfrozen bits, and the channels with low reliability in each sub-polar code cause more errors when transmitting bits, so the second step of constructing the critical set is to add the index of the channels with low reliability.

After the bits with low PW values are gradually added to the critical set composed of original MHW and sub-MHW bits, the *BLER* of the SCL decoding algorithm is shown in Fig.5. Simulated polar codes are at code length N=1 024, code rate R=1/2, list size L=8 and *SNR*=2.75 dB.

From Fig.5, as the number of low-reliability channels added to the critical set increases, the *BLER* gradually decreases but eventually tends to a stable value. Therefore, variable a is introduced to control the added number of low-reliability channels. After containing channels with MHW or sub-MHW to the critical set, continue to add a% channels with low PW value. a can be flexibly selected according to the target *BLER*. If high error correction performance is required, add the channels with a minimum percentage low PW value to the critical set if there is almost no loss to *BLER* compared to the conventional SCL decoding algorithm (for example, a equals 20 for code length N=1 024), but the complexity reduction is less by this way. On the contrary, if the error correction performance can be lower, a% channels with low PW value are added to the critical set according to the target *BLER* (for example, a equals 15 for code length N=1 024). Although the computational complexity reduction is larger, there is a significant loss in error correction performance.

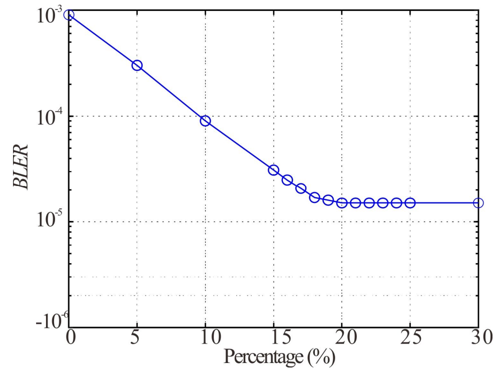

Fig.5 *BLER* performance after adding low-reliability bits index to the critical set

The proposed construction scheme of the critical set is described as follows.

(1) Add the MHW or sub-MHW channel index into the CS.

Calculate the number of MHW and sub-MHW bits and determine the corresponding channel index of each sub-polar code.

If the number of the MHW bits is more than 1% of the bits in the unfrozen set, only add the MHW channel index into the CS. Contrarily, the index of the sub-MHW bits is additionally added to the critical set.

(2) Add a% of the channel index with the low PW values into the CS according to the target *BLER*.

The proposed CS-SCL is obtained by combining the critical set and the principle of path reduction. The flow chart of CS-SCL coding and decoding is shown in Fig.6.

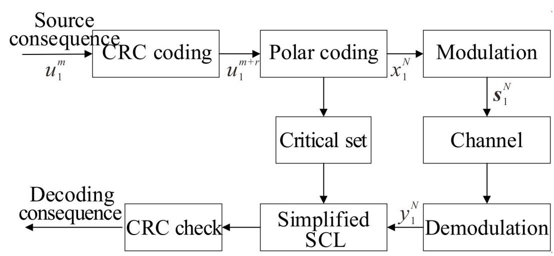

Fig.6 Flow chart of CA-SCL coding and decoding

Assume that the polar code length is N and the number of added CRC codes is r. The sequence of transmitted information bits u_1^m is coded to obtain u_1^{m+r} after the CRC encoder. Finally, the polar code u_1^{m+r} is coded into the polar code to obtain the codeword x_1^N. Then the transmission signal vector $\boldsymbol{s}_1^N$ is generated by applying binary phase-shift keying (BPSK) on codeword x_1^N, and $\boldsymbol{s}_1^N$ is sent through the additive white Gaussian noise

(AWGN) channel. The received sequence $y_1^N=(y_1,y_2,...,y_N)$ can be expressed as $y_i=s_i+n_i$, where $n_i \sim N(0,\sigma^2)$. The specific decoding steps are shown in Tab.1.

Tab.1 Simplified SCL decoding algorithm based on the critical set to reduce path splitting

input: received consequence, critical set;
output: decoding consequence u_1^N;
1: **for** i = 1: N
2: **if** $i \in CS$
3: decode u_i according to the SCL decoding algorithm above;
4: **else**
5: perform hard decision via Eq.(2);
6: **end if**
7: **end for**
8: If the CRC is available, the decoding result passes the CRC check is selected to output, otherwise the decoding result with the smallest PM will be output.

The number of bits that needs path copying, sorting, and deleting during decoding, i.e., bits within the critical set, is analyzed as the complexity. The computational complexity of the conventional SCL decoding algorithm, the proposed CS-SCL decoding algorithm, the PS-SS-SCL(1) algorithm and PS-SS-SCL(2) algorithm proposed in Ref.[12] are calculated at the code lengths of 1 024 and 512, respectively, as shown in Tab.2. Among them, the PS-SS-SCL(1) algorithm constructs the critical set by selecting the first bit of all Rate-1 nodes, and the PS-SS-SCL(2) algorithm constructs the critical set by selecting the first and second bits of Rate-1 nodes.

As Tab.2 shows that compared with the traditional SCL algorithm, the computational complexity of the proposed CS-SCL algorithm is reduced by at least 70% at N=1 024 and 512. Compared with PS-SS-SCL(2) algorithm, which has less error correction performance loss, the computational complexity of the CS-SCL algorithm is also reduced by 6.1% and 6.2% at N=1 024 and 512, respectively. It can be concluded that by combining the computational complexity and the performance of error correction, the proposed algorithm can effectively reduce the computational complexity in decoding with a minor loss of error correction performance. Moreover, because the value of the introduced variable a can be selected flexibly, the computational complexity can be further reduced at the price of the loss for more error correction performance.

In order to verify the superiority of the proposed CS-SCL decoding algorithm, all simulations are under the construction method of polarization weights, BPSK modulation, AWGN channel, code rate R=1/2, list size L=8, and 16-bit CRC with the generator polynomial $g(x)=x^{16}+x^{12}+x^6+x^5+x+1$.

Tab.2 Computational complexity of four algorithms

	N=1 024		N=512	
Algorithm	Critical set	Complexity (%)	Critical set	Complexity (%)
SCL	528	100	272	100
PS-SS-SCL(1)	108	20.5	62	22.8
PS-SS-SCL(2)	170	30.2	98	36.0
CS-SCL	127	24.1	81	29.8

Fig.7 shows the simulation results of the error correction performance comparison of the four algorithms at N=1 024 and a=20. As can be seen from Fig.7, the proposed algorithm has the smallest error correction performance loss compared with the conventional SCL decoding algorithm. At the code length of 1 024 and $BLER$=10^{-5}, the PS-SS-SCL(1) algorithm and the PS-SS-SCL(2) algorithm have a performance loss of 0.12 dB and 0.07 dB, respectively. The proposed CS-SCL algorithm has a loss of about 0.02 dB, which improves the gain by 0.05 dB compared with the PS-SS-SCL(2) algorithm.

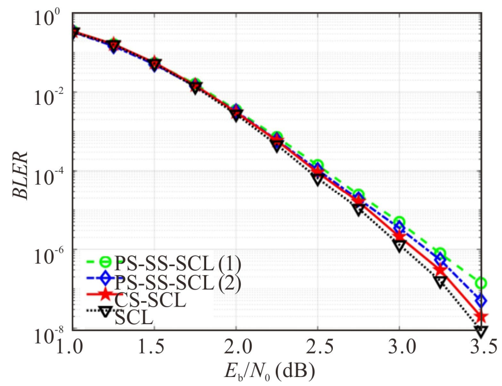

Fig.7 *BLER* under different decoding schemes at *N*=1 024 and *a*=20

Fig.8 shows the error correction performance of the four algorithms for N=512. The loss of error correction performance for the proposed algorithm is the least compared with the conventional SCL decoding algorithm. At $BLER$=10^{-5}, the PS-SS-SCL(1) algorithm and the PS-SS-SCL(2) algorithm have relatively large error correction performance losses of 0.25 dB and 0.13 dB, respectively, while the proposed CS-SCL decoding algorithm has almost no loss and improves the gain by 0.13 dB compared to PS-SS-SCL(2) algorithm.

Additionally, in order to further verify the effectiveness of the proposed algorithm, the AWGN channel is replaced with the Rayleigh fading channel through the complex Gaussian method[16], and the other simulation conditions remain unchanged. The error correction performance of the four algorithms is compared at N=512 and a=23 as shown in Fig.9. It can be concluded that at

BLER=10^{-5}, compared with the conventional SCL decoding algorithm, the PS-SS-SCL(1) algorithm and the PS-SS-SCL(2) algorithm have about 0.4 dB and 0.2 dB performance losses, respectively, while the error performance loss of the proposed algorithm is minor with less than 0.1 dB.

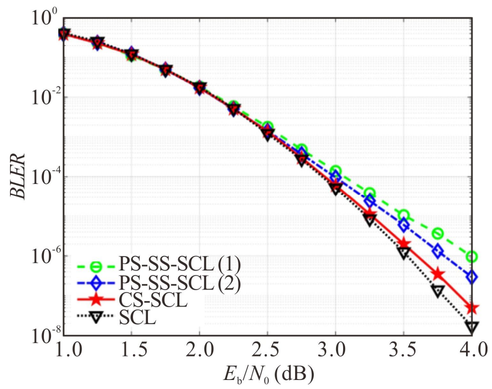

Fig.8 *BLER* under different decoding schemes at *N*=512 and *a*=23

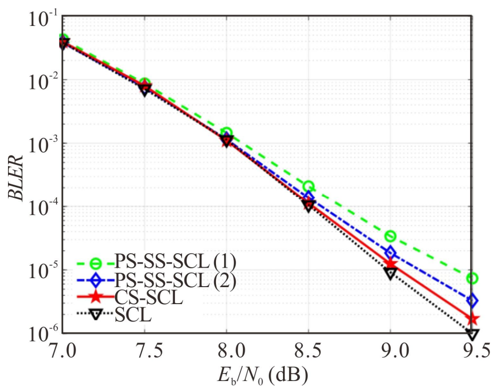

Fig.9 *BLER* under different decoding schemes at *N*=512 over Rayleigh fading channel

In order to reduce the high computational complexity of the conventional SCL decoding algorithm for polar codes, a simplified SCL decoding algorithm based on the critical set to reduce the number of path splitting is proposed. The simulation results show that the proposed algorithm has minimal loss of error correction performance for both short and moderate code lengths, and significantly reduces the computational complexity of the SCL decoding. Thus, the proposed CS-SCL decoding algorithm is effective and can provide a better compromise selection between decoding performance and complexity for the decoding algorithm of polar codes.

Ethics declarations

Conflicts of interest

The authors declare no conflict of interest.

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